MATH4210: Financial Mathematics Tutorial 5

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Convergence of r.v.s

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. X and $\{X_n\}$ are $\mathbb{R}$ valued (sequence of) r.v.s.

Definition (Convergence almost surely)

Denote by $X_n \to X$ a.s. (almost surely) if

$$\mathbb{P}[\{\omega \in \Omega : \lim_{n \to \infty} X_n(\omega) = X(\omega)\}] = 1$$

Definition (Convergence in Probability)

Denote by $X_n \to X$ in probability if for any $\rho > 0$

$$\lim_{n \to \infty} \mathbb{P}[\{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \rho\}] = 0$$

Proposition proved

$X_n \to X$ a.s. implies $X_n \to X$ in probability.

Proposition admit (Search Borel – Cantelli Lemma)

$X_n \to X$ in probability implies there exists a subsequence of X_n converging to X a.s..

Definition (Convergence in Law (in Distribution))

Let F_n and F be the c.d.f. of X_n and X for all $n \in \mathbb{N}$. $X_n \to X$ in Law (in Distribution) if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for any $x \in \mathbb{R}$ where F is continuous at x.

Proposition admit

$X_n \to X$ in probability implies $X_n \to X$ in Law.

Convergence of r.v.s.

Definition

Given $p > 0$, denote by $X_n \to X$ in L^p if

$\lim_{n \to +\infty} \int_{\mathbb{R}} |f_n - f|^p(x)\,dx = 0$

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^p] = 0$$

Question

(a). Show that $X_n \to X$ in L^2 implies $X_n \to X$ in L^1.
(b). Show that $X_n \to X$ in L^p implies $X_n \to X$ in probability.

(a). proof : $\left(\lim_{n\to+\infty} \mathbb{E}[|X_n - X|^2] = 0.\right) \Rightarrow \left(\lim_{n\to+\infty} \mathbb{E}[|X_n - X|] = 0\right)$.

Assume $\swarrow$ is true.

Fix $n \in \mathbb{N}$

$\mathbb{E}[|X_n - X|] \leq$ " $f(\mathbb{E}[|X_n - X|^2])$ "

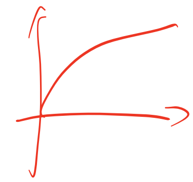
$\mathbb{E}[|X_n - X|] = \mathbb{E}[\sqrt{|X_n - X|^2}]$

$\leq \sqrt{\mathbb{E}[|X_n - X|^2]}$ ↙ Jensen's inequality.

$\to 0.$

if f is convex. then $f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))$

if f is concave then: $\mathbb{E}(f(X)) \leq f(\mathbb{E}(X))$



Since $x \mapsto \sqrt{x}$ is continuous

and $\lim_{n \to +\infty} \mathbb{E}[|X_n - X|^2] = 0$ $\square$

(b). <u>proof</u> :

Fix $\rho > 0$. fix $n \in \mathbb{N}$.

$\mathbb{P}[\{\omega \in \Omega : |X_n - X| \geq \rho\}] \xrightarrow{n \to +\infty} 0$

$= \mathbb{P}[\{|X_n - X|^p \geq \rho^p\}]$

$\leq \frac{1}{\rho^p} \mathbb{E}[|X_n - X|^p]$ → by Markov's inequality

$\mathbb{P}[\{|X| > M\}] \leq \frac{1}{M} \mathbb{E}[|X|]$

$\to 0$ as $n \to +\infty$.

by $\mathbb{L}^p$ convergence. $\square$

Montone CV theorem

Dominated CV theorem

$X_n \xrightarrow{a.s.} X$ $\xRightarrow{\text{Conditions}}$ $X_n \to X$ in $\mathbb{L}^p$.

$\Downarrow$ $\Uparrow$ subsequence

$X_n \xrightarrow{\mathbb{P}} X$ $\Leftarrow$ ($X_n \to X$ in $\mathbb{L}^p$)

$\Downarrow$

$X_n \xrightarrow{\mathcal{L}} X$

Question

Let $f : \mathbb{R} \to \mathbb{R}$ *be such that for some constant* $C > 0$, $f(x) < e^{C|x|}$ *for all* $x \in \mathbb{R}$. *Define*

$$u(t, x) = \mathbb{E}[f(B_T)|B_t = x] = \mathbb{E}[f(B_T - B_t + x)].$$

Show that

(a)

$$\partial_x u(t, x) = \mathbb{E}[\frac{B_T - B_t}{T - t} f(B_T - B_t + x)]$$

(b)

$$\partial_x^2 u(t, x) = \mathbb{E}[\frac{(B_T - B_t)^2 + (T - t)}{(T - t)^2} f(B_T - B_t + x)]$$

$$= \mathbb{E}[\frac{Z^2 + (T-t)}{(T-t)^2} f(Z+x)], \quad Z \sim N(0, T-t).$$

(a). <u>Proof</u>: Fix $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

$u(t,x) = \mathbb{E}[f(B_T - B_t + x)]$

Denote $Z = N(0, T-t)$, denote by p_Z its pdf.

So $u(t,x) = \mathbb{E}[f(Z + x)]$.

$$= \int_{\mathbb{R}} f(y+x)\, p_Z(y)\, dy.$$

Then $\partial_x u(t,x) = \int_{\mathbb{R}} f'(y+x)\, p_Z(y)\, dy$.

IBP.
$$= \underbrace{\Big[f(y+x)\, p_Z(y)\Big]_{y\to-\infty}^{y\to+\infty}}_{\Downarrow\ 0} - \int_{\mathbb{R}} f(y+x)\, p_Z'(y)\, dy.$$

$$p_Z'(y) = \frac{1}{\sqrt{2\pi(T-t)}}\, e^{-\frac{y^2}{2(T-t)}} \cdot \left(-\frac{y}{(T-t)}\right)$$

So $\partial_x u(t,x) = 0 + \frac{1}{T-t} \int_{\mathbb{R}} y f(y+x)\, p_Z(y)\, dy$

$$= \mathbb{E}\left[\frac{f(Z+x)}{T-t} \cdot Z\right]$$

$$= \mathbb{E}\left[\frac{f(B_T - B_t + x)}{T-t}(B_T - B_t)\right] \qquad \square$$

(b). Similarly, we apply the IBP . . . .

Greeks of Option

$$d_1 : x \mapsto \frac{\ln(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

$$C_E : (t, x) \mapsto x N(d_1(x)) - e^{-r(T-t)} K N(d_2(x))$$

Question

Consider the European call option price at time t:

$$C_E(t, S_t) = S_t N(d_1) - e^{-r(T-t)} K N(d_2)$$

where $d_1 = \frac{\ln(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$ *and* $d_2 = d_1 - \sigma\sqrt{T-t}$. *Compute*
(a) Delta: $\Delta = \partial_x C_E(t, S_t)$.
(b) Gamma: $\Gamma = \partial_x^2 C_E(t, S_t)$.

(a). $d_1(S_t) = \frac{\ln(S_t/K) + (r+\sigma^2/2)\cdot\tau}{\sigma\sqrt{\tau}}$, $\tau = T-t$.

$$\partial_x d_1(S_t) = \frac{1}{\sigma\sqrt{\tau}} \cdot \frac{1}{S_t} = \partial_x d_2(S_t)$$

$$N(d_1(S_t)) = \int_{-\infty}^{d_1(S_t)} f_Z(y)\,dy,$$ where $Z \sim N(0,1)$
f_Z its pdf.

So. $\frac{\partial}{\partial x} N(d_1(S_t)) = f_Z(d_1(S_t)) \cdot \partial_x d_1(S_t)$.

Therefore by chain rule,

$$\partial_x C_E(t, S_t) = N(d_1(S_t)) + S_t \cdot f_Z(d_1(S_t)) \cdot \partial_x d_1(S_t) - e^{\tau} K \cdot f_Z(d_2(S_t)) \cdot \partial_x d_2(S_t)$$

$$= N(d_1(S_t)) + \partial_x d_1(S_t) \cdot \left(S_t f_Z(d_1(S_t)) - e^{\tau} K f_Z(d_2(S_t))\right)$$

$$= N(d_1(S_t)) + \partial_x d_1(S_t) \cdot \frac{1}{\sqrt{2\pi}\sigma}\left(S_t \cdot \underline{e^{\frac{-d_1^2}{2\sigma^2}}} - e^{-r\tau} K e^{-\frac{d_2^2}{2\sigma^2}}\right)$$

Note that.

$$d_1^2 - d_2^2 = (d_1 + d_2)(d_1 - d_2)$$

$$= \left(2\frac{\ln(S_t/K) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau}\right) \cdot \left(\sigma \cdot \sqrt{\tau}\right)$$

$$= 2\ln(S_t/K) + 2r\tau$$

$$\text{So } \partial_x C_E(t, S_t) = N(d_1(S_t)) + \frac{\partial_x d_1(S_t)}{\sqrt{2\pi}} \cdot e^{\frac{-d_1^2}{2}}\left(S_t - e^{r\tau} K e^{-\frac{d_2^2 - d_1^2}{2}}\right)$$

$$S_t - e^{-r\tau} \cdot K \cdot e^{-\frac{d_2^2 - d_1^2}{2}} = 0.$$

(A) a consequence

$$\partial_x C_E(t, S_t) = N(d_1). \qquad \square$$

(b). $\Gamma = \partial_x^2 C_E(t, S_t)$

$$= \partial_x N(d_1(S_t))$$

$$= f_Z(d_1(S_t)) \cdot \partial_x d_1(S_t).$$

$$= \frac{1}{\sigma\sqrt{\tau} S_t} f_Z(d_1(S_t)).$$